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The quantum mechanics of the $N=2$ extended supersymmetric nonlinear σ-model

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Abstract. The classical and quantum mechanical formalisms of the model are developed. The quantisation is performed in such a way that the quantum theory can be represented explicitly in as simple a form as possible, and the problem of ordering of operators is resolved so as to maintain the extended $N = 2$ supersymmetry algebra of the classical theory.

1. Introduction

In this paper we study the extended $N=2$ supersymmetric nonlinear σ-model in one (time) dimension. Our motivation for this work has been discussed in a previous study (Davis *et al* 1984) where analogous results for the $N=1$ models have been presented. We therefore limit our presentation to that of results and discussion of those aspects of the $N=2$ theory that differ from the $N=1$ theory. Since the base manifold in the former is a Kahler manifold, as compared with a Riemann manifold in the latter, the calculations are of a somewhat different nature. In the $N=1$ quantum theory, the hermiticity was sufficient to resolve the order of operator problem for the supercharges and, through the corresponding supersymmetry algebra, that of the Hamiltonian. Here the same is achieved by demanding that the derived $N=2$ supersymmetry algebra is satisfied by the supercharges and Hamiltonian and the obvious classical limits obtained.

The paper is organised as follows. After a brief description in § 2 of the derivation of the classical supersymmetric Lagrangian and supercharges we proceed in § 3 to derive the classical canonical formalism, using the Dirac bracket formalism to take account of the second class constraints of the theory. We observe that a change of variables allows the formalism to exhibit the geometrical structure of the theory. In § 4 the canonical formalism is expressed in terms of a new set of variables, which has the property of decoupling the Grassmann variables from the rest and 'diagonalising' the Dirac brackets, thus allowing a simple representation in the quantum theory. In § 5 the fundamental quantum commutation relationships are written down for the three sets of variables introduced, with the operator ordering ambiguities resolved. Finally in § 6 we derive the quantum mechanical supercharges and Hamiltonian with their problem of the order of operators also completely resolved.

2. The classical $N=2$ theory

The Lagrangian of a nonlinear σ-model with N complex scalar fields A^i taking values

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in a complex Kahler manifold can be written in one (time) dimension as

$$L = h_{ij}(A, \bar{A})\, \partial_t A^i\, \partial_t \bar{A}^j. \tag{2.1}$$

Here the bar denotes complex conjugation and h_{ij} is the Kahlerian metric of the manifold. In fact (Zumino 1979, Alvarez-Gaume and Freedman 1981) the construction of the $N=2$ supersymmetric theory requires that the non-linear σ-model is defined on a Kahler manifold.

The Kahler property requires the existence of a potential $V(A, \bar{A})$ (Alvarez-Gaume and Freedman 1980, Flaherty 1976, Zumino 1979) such that

$$h_{ij} = 2\partial^2 V(A, \bar{A})/\partial A^i\, \partial \bar{A}^j. \tag{2.2}$$

In terms of V the Lagrangian of the supersymmetric extension of the theory is (Zumino 1979)

$$L = \tfrac{1}{8}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} D_\alpha \bar{D}_\gamma \bar{D}_\delta D_\beta V(\phi, \bar{\phi})|_0, \tag{2.3}$$

where the D_α are the supercovariant derivatives, ϕ^i the chiral superfields defined by $\bar{D}\phi = D\bar{\phi} = 0$ and the notation '$|_0$' indicates that values at $\theta = \bar{\theta} = 0$ be taken. The independent complex fields contained in ϕ are given by

$$A^i = \phi^i|_0, \qquad \Psi_\alpha = 2^{-1/2} D_\alpha \phi^i|_0, \qquad F^i = \tfrac{1}{4}\varepsilon_{\alpha\beta} D_\alpha D_\beta \phi^i|_0. \tag{2.4}$$

The Lagrangian (2.3) can be evaluated in terms of these fields. Upon use of their equations of motion to eliminate the auxiliary fields F^i and $\bar{F}^j$, we obtain (Zumino 1979)

$$\begin{aligned} L = {} & h_{ij}\, \partial_t A^i\, \partial_t \bar{A}^j + \tfrac{1}{2}\mathrm{i} h_{ij}(\bar{\psi}^j{}_\alpha \partial_t \psi^i{}_\alpha - \partial_t \bar{\psi}^j{}_\alpha \psi^i{}_\alpha) \\ & + \frac{\mathrm{i}}{2}\left(\frac{\partial h_{ij}}{\partial A^k}\, \partial_t A^k - \frac{\partial h_{ij}}{\partial \bar{A}^k}\, \partial_t \bar{A}^k\right) \bar{\psi}^j{}_\alpha \psi^i{}_\alpha - \tfrac{1}{4} k_{ijkl}(\bar{\psi}^l{}_\alpha \varepsilon_{\alpha\beta} \bar{\psi}^j{}_\beta)(\psi^i{}_\gamma \varepsilon_{\gamma\delta} \psi^k{}_\delta) \end{aligned} \tag{2.5}$$

where the curvature tensor of the manifold is given by

$$k_{ijkl} = \frac{\partial^2 h_{kl}}{\partial \bar{A}^j\, \partial A^i} - \frac{\partial h_{ml}}{\partial \bar{A}^j} h^{mn} \frac{\partial h_{kn}}{\partial A^i},$$

and h^{mn} is the inverse of the metric tensor, i.e. $h^{mn} h_{kn} = h^{nm} h_{nk} = \delta^m_k$.

The supercharges Q_α and $\bar{Q}_\alpha$ are found by considering a supersymmetry transformation of the form $\phi^i \to \phi^i + \delta\phi^i$ with complex Grassman parameters ε_α, and Noether's theorem written in the form

$$(\bar{\varepsilon}_\alpha \bar{Q}_\alpha + Q_\alpha \varepsilon_\alpha) = \left(\delta A^i \frac{\partial L}{\partial \partial_t A^i} + \delta \psi^i{}_\alpha \frac{\partial L}{\partial \partial_t \psi^i{}_\alpha} - K\right) + (\text{Hermitian conjugate}). \tag{2.6}$$

K is obtained from the variation $\delta V(\phi, \bar{\phi})$ of the potential under the supersymmetry transformation via

$$\partial_t (K + \bar{K}) = \delta L = \tfrac{1}{8}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} D_\alpha \bar{D}_\gamma \bar{D}_\delta D_\beta \delta V(\phi, \bar{\phi})|_0.$$

Using $\delta A^i = \delta\phi^i|_0$ and $\delta\psi^i{}_\alpha = 2^{-1/2} D_\alpha \delta\phi^i|_0$, etc, we are led to

$$Q_\alpha = 2^{1/2} \varepsilon_{\alpha\beta} \psi^i{}_\beta h_{ij}\, \partial_t \bar{A}^j, \qquad \bar{Q}_\alpha = 2^{1/2} \varepsilon_{\alpha\beta} \bar{\psi}^j{}_\beta h_{ij}\, \partial_t A^i. \tag{2.7}$$

3. Classical canonical formulations

From (2.5) we find for the conjugate momenta

$$P_i = \frac{\partial L}{\partial \partial_t A^i} = h_{ij}\, \partial_t \bar{A}^j + \frac{i}{2}\frac{\partial h_{kj}}{\partial A^i}\bar{\psi}^j{}_\alpha \psi^k{}_\alpha,$$
$$\bar{P}_i = \frac{\partial L}{\partial \partial_t \bar{A}^i} = h_{ji}\, \partial_t A^j - \frac{i}{2}\frac{\partial h_{kj}}{\partial \bar{A}^i}\bar{\psi}^j{}_\alpha \psi^k{}_\alpha, \tag{3.1}$$

$$\tau_{i\alpha} = \frac{\partial L}{\partial \partial_t \psi^i{}_\alpha} = -\frac{i}{2} h_{ij}\bar{\psi}^j{}_\alpha, \qquad \bar{\tau}_{i\alpha} = \frac{\partial L}{\partial \partial_t \bar{\psi}^i{}_\alpha} = -\frac{i}{2} h_{ji}\psi^j{}_\alpha. \tag{3.2}$$

We see from (3.1) that the canonical formulation leads to constraints

$$\chi_{i\alpha} = \tau_{i\alpha} + \tfrac{1}{2} i\, h_{ij}\bar{\psi}^j{}_\alpha = 0, \qquad \bar{\chi}_{i\alpha} = \bar{\tau}_{i\alpha} + \tfrac{1}{2} i\, h_{ij}\psi^i{}_\alpha = 0. \tag{3.3}$$

These are second class constraints (Dirac 1964) and are the only constraints in the model. They can be taken into account by using the Dirac (1964) formalism suitably generalised (Casalbuoni 1976) to include anticommuting quantities.

The non-vanishing Poisson brackets for the theory are

$$\{P_i, A^j\} = -\delta^j_i, \qquad \{\tau_{i\alpha}, \psi^j{}_\beta\} = -\delta^j_i \delta_{\alpha\beta}, \tag{3.4a, b}$$
$$\{\chi_{i\alpha}, \bar{\chi}_{j\beta}\} = -i h_{ij}\delta_{\alpha\beta}, \tag{3.4c}$$

plus the corresponding ones for the complex conjugate variables. For any two fields A and B the Dirac bracket, $\{A, B\}^*$, is defined in this theory to be

$$\{A, B\}^* = \{A, B\} - i\{A, x_{i\alpha}\}h^{ij}\{\bar{x}_{j\alpha}, B\} - i\{A, \bar{x}_{j\alpha}\}h^{ij}\{x_{i\alpha}, B\} \tag{3.5}$$

Within the Dirac bracket formalism the constraints can be set equal to zero since the Dirac bracket of any field with any one of the constraints vanishes (Dirac 1964).

The non-vanishing Dirac brackets for the independent fields are

$$\{P_i, A^j\}^* = -\delta^j_i, \qquad \{\psi^i{}_\alpha, \bar{\psi}^j{}_\beta\}^* = -i h^{ij}\delta_{\alpha\beta} \tag{3.6a, b}$$

$$\{P_i, P_j\}^* = -\frac{i}{4}h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial A^j} - \frac{\partial h_{mk}}{\partial A^j}\frac{\partial h_{ln}}{\partial A^i}\right)\bar{\psi}^k{}_\alpha \psi^l{}_\alpha, \tag{3.6c}$$

$$\{P_i, \bar{P}_j\}^* = -\frac{i}{4}h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial \bar{A}^j} - \frac{\partial h_{mk}}{\partial \bar{A}^j}\frac{\partial h_{ln}}{\partial A^i}\right)\bar{\psi}^k{}_\alpha \psi^l{}_\alpha, \tag{3.6d}$$

$$\{P_i, \psi^j{}_\alpha\}^* = \tfrac{1}{2}h^{jm}(\partial h_{km}/\partial A^i)\psi^k{}_\alpha, \tag{3.6e}$$

$$\{P_i, \bar{\psi}^j{}_\alpha\}^* = \tfrac{1}{2}h^{mj}(\partial h_{mk}/\partial A^i)\bar{\psi}^k{}_\alpha, \tag{3.6f}$$

plus those involving the complex conjugate variables.

The Hamiltonian H is given by

$$H = \pi_i h^{ij}\bar{\pi}_j + \tfrac{1}{4}k_{ijkl}\bar{\psi}^l{}_\alpha \varepsilon_{\alpha\beta}\bar{\psi}^j{}_\beta \psi^i{}_\gamma \varepsilon_{\gamma\delta}\psi^k{}_\delta \tag{3.7}$$

where

$$\pi_i = P_i - \frac{i}{2}\frac{\partial h_{mn}}{\partial A^i}\bar{\psi}^n{}_\alpha \psi^m{}_\alpha, \qquad \bar{\pi}_i = \bar{P}_i + \frac{i}{2}\frac{\partial h_{mn}}{\partial A^i}\bar{\psi}^n{}_\alpha \psi^m{}_\alpha. \tag{3.8}$$

If a change of variables is made from the P_i and $\bar{P}_i$ to the π_i and $\bar{\pi}_i$ using (3.8), the Dirac brackets (3.6) are expressible in terms of the geometrical properties of the theory as follows:

$$\{\pi_i, A^j\}^* = -\delta^j_i, \qquad \{\pi_i, \psi^j{}_\alpha\}^* = \Gamma^j_{ik}\psi^k{}_\alpha, \tag{3.9a, b}$$

$$\{\pi_i, \bar{\psi}^j{}_\alpha\}^* = 0, \qquad \{\pi_i, \pi_j\}^* = 0, \tag{3.9c, d}$$

$$\{\pi_i, \bar{\pi}_j\}^* = -\mathrm{i}k_{ijkl}\bar{\psi}^l{}_\alpha\psi^k{}_\alpha, \qquad \{\psi^i{}_\alpha, \bar{\psi}^j{}_\beta\}^* = -\mathrm{i}h^{ij}\delta_{\alpha\beta}. \tag{3.9e, f}$$

Here $\Gamma^j_{ik} = h^{jm}\partial h_{im}/\partial A^k$ are the components of the connection.

The transformations (3.8) are not unique since if a term $B\Gamma^l_{ij}$ (B real and arbitrary) is added to the RHS of (3.8a) and its complex conjugate to (3.8b), the Dirac brackets (3.9) are unchanged.

Using (2.7), (3.1) and (3.8) it follows that the supercharges can be written in the form

$$Q_\alpha = 2^{1/2}\varepsilon_{\alpha\beta}\psi^i{}_\beta\pi_i, \qquad \bar{Q}_\alpha = 2^{1/2}\varepsilon_{\alpha\beta}\bar{\psi}^j{}_\beta\bar{\pi}_j. \tag{3.10}$$

The above results show that the variables π^i and $\bar{\pi}^i$ lead to simpler expressions for the Hamiltonian and supercharges which is convenient for canonical calculations. In addition, we will see later that the simplicity of the expressions for the supercharges allows for a straightforward resolution of the operator ordering problem for these variables in the quantum theory.

Using the relevant Dirac brackets it is easily shown that the supercharges, expressed in either of the two sets of variables introduced so far, obey the expected canonical supersymmetry algebras

$$\{Q_\alpha, Q_\beta\}^* = \{\bar{Q}_\alpha, \bar{Q}_\beta\}^* = 0, \qquad \{Q_\alpha, \bar{Q}_\beta\}^* = -2\mathrm{i}H\delta_{\alpha\beta}. \tag{3.11a,b}$$

4. Simplified canonical equation

We require a set of variables that decouples the Grassmann variables from the rest. To achieve this we introduce a vielbein transformation

$$\lambda^a{}_\alpha = e^a_i\psi^i{}_\alpha, \qquad \bar{\lambda}^a{}_\alpha = \bar{e}^a_i\bar{\psi}^i{}_\alpha, \tag{4.1}$$

where the vielbeins and their inverses are defined by $h_{ij} = e^a_i\bar{e}^b_j\eta_{ab}$, $h^{ij} = e^i_a\bar{e}^j_b\eta^{ab}$, $e^a_ie^i_b = \delta^a_b$, $e^i_ae^a_j = \delta^i_j$, and η_{ab} is the locally flat metric. Using (4.1) and (3.9) we deduce

$$\{\lambda^a{}_\alpha, \bar{\lambda}^b{}_\beta\}^* = -i\eta^{ab}\delta_{\alpha\beta}, \qquad \{\lambda^a{}_\alpha, \bar{\lambda}^b{}_\beta\}^* = 0, \tag{4.2a, b}$$

$$\{\lambda^a{}_\alpha, A^i\}^* = \{\bar{\lambda}^a{}_\alpha, A^i\}^* = 0, \qquad \{\lambda^a{}_\alpha, \pi_i\}^* = \omega^a{}_{ib}\lambda^b{}_\alpha, \tag{4.2c, d}$$

$$\{\bar{\lambda}^a{}_\alpha, \pi_i\}^* = \omega_{\bar{b}i}{}^{\bar{a}}\bar{\lambda}^b{}_\alpha \tag{4.2e}$$

where the spin connections are defined by

$$D_ie^a_j = e^a_{j,i} - \omega^a{}_{ib}e^b_j - \Gamma^k_{ij}e^a_k = 0, \tag{4.3a}$$

$$D_i\bar{e}^a_j = \bar{e}^a_{j,i} - \omega_{\bar{b}i}{}^{\bar{a}}\bar{e}^b_j = 0. \tag{4.3b}$$

The notation $\partial e^b{}_k/\partial A^i = e^b_{k,i}$, etc, and the fact that the only non-vanishing components of the connection are those with all indices either 'holomorphic' or 'antiholomorphic' (Alvarez-Gaume and Freedman 1980, Flaherty 1976) has been used.

Relations (4.2d) and (4.2e) motivate the change of variables

$$R_i = \pi_i + \mathrm{i}\omega_{\bar{b}ia}\bar{\lambda}^b{}_\alpha\lambda^a{}_\alpha \tag{4.4}$$

so that

$$\{R_i, \lambda^a{}_\alpha\}^* = \{R_i, \bar{\lambda}^a{}_\alpha\}^* = 0, \tag{4.5a}$$

thus decoupling the Grassmann variables from the rest. The other Dirac brackets involving the R_i are

$$\{R_i, R_j\}^* = \{R_i, \bar{R}_j\}^* = 0, \qquad \{R_i, A^j\}^* = -\delta^j_i \tag{4.5b, c}$$

$$\{R_i, \bar{A}^j\}^* = 0. \tag{4.5d}$$

In terms of these variables the supercharges are

$$Q_\alpha = 2^{1/2}\varepsilon_{\alpha\beta}e^i_d\lambda^d{}_\beta(R_i - \mathrm{i}\omega_{\bar{b}ia}\bar{\lambda}^b_\gamma\lambda^a_\gamma), \qquad \bar{Q}_\alpha = 2^{1/2}\varepsilon_{\alpha\beta}\bar{e}^i_d\bar{\lambda}^d{}_\beta(\bar{R}_i + \mathrm{i}\omega_{\bar{a}ib}\bar{\lambda}^b_\gamma\lambda^a_\gamma), \tag{4.6}$$

and they obey the expected canonical algebra (3.11).

The important property of (4.2) and (4.5) is that the 'diagonal' structure of the Dirac brackets allows an explicit representation of the corresponding quantum mechanical algebra. The bosonic sector is represented in terms of differential operators with respect to the fields A^i and the fermionic sector with creation and annihilation operators or their matrix equivalent.

5. Quantisation

The quantum mechanical theory is obtained by replacing the Dirac brackets by $-\mathrm{i}$ times the appropriate quantum bracket and taking $\chi_i = 0$ in the Hamiltonian.

We consider in turn the three sets of variables previously introduced and write down the fundamental quantum commutators and anticommutators for each set of variables.

(i) Variables A^i, P^i, ψ^i

Using (3.6) and the above prescription we obtain

$$[P_i, A^j] = -\mathrm{i}\delta^j_i, \qquad \{\psi^i{}_\alpha, \bar{\psi}^j{}_\beta\} = h^{ij}\delta_{\alpha\beta}, \tag{5.1a, b}$$

$$[P_i, P_j] = \tfrac{1}{4}h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial A^j} - \frac{\partial h_{mk}}{\partial A^j}\frac{\partial h_{ln}}{\partial A^i}\right)\bar{\psi}^k{}_\alpha\psi^l{}_\alpha, \tag{5.1c}$$

$$[P_i, \bar{P}_j] = \tfrac{1}{4}h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial \bar{A}^j} - \frac{\partial h_{mk}}{\partial \bar{A}^j}\frac{\partial h_{ln}}{\partial A^i}\right)\bar{\psi}^k{}_\alpha\psi^l{}_\alpha, \tag{5.1d}$$

$$[P_i, \psi^j{}_\alpha] = \frac{\mathrm{i}}{2}h^{jm}\frac{\partial h_{km}}{\partial A^i}\psi^k{}_\alpha, \qquad [P_i, \bar{\psi}^j{}_\alpha] = \frac{\mathrm{i}}{2}h^{mj}\frac{\partial h_{mk}}{\partial A^i}\bar{\psi}^k{}_\alpha. \tag{5.1e, f}$$

In (5.1) there is no operator ordering ambiguity, not even in (5.1d) and (5.1e) where the two possible orderings of the fermionic fields lead to expressions that differ by a term proportional to

$$h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial A^j} - \frac{\partial h_{mk}}{\partial A^j}\frac{\partial h_{ln}}{\partial A^i}\right)h^{lk}$$

and

$$h^{mn}\left(\frac{\partial h_{mk}}{\partial A^i}\frac{\partial h_{ln}}{\partial \bar{A}^j}-\frac{\partial h_{mk}}{\partial \bar{A}^j}\frac{\partial h_{ln}}{\partial A^i}\right)h^{lk}$$

respectively. Both of these terms vanish identically as is verified by a relabelling of the dummy indices.

(ii) Variables A^i, π^i, $\psi^i{}_\alpha$

From (3.9) we deduce

$$[\pi_i, A^j]=[\bar{\pi}_i, \bar{A}^j]=-\mathrm{i}\delta^j_i, \qquad \{\psi^i{}_\alpha, \bar{\psi}^j{}_\beta\}=h^{ij}\delta_{\alpha\beta}, \tag{5.2a, b}$$

$$[\pi_i, \psi^j{}_\alpha]=\mathrm{i}\Gamma^j{}_{ik}\psi^k{}_\alpha, \qquad [\bar{\pi}_i, \bar{\psi}^j{}_\alpha]=\mathrm{i}\bar{\Gamma}^j_{ik}\bar{\psi}^k{}_\alpha, \tag{5.2c, d}$$

$$[\pi_i, \bar{\pi}_j]=k_{ijkl}\bar{\psi}^l{}_\alpha\psi^k{}_\alpha-2B\Gamma^k_{ki,\bar{j}}. \tag{5.2e}$$

Only in (5.2e) is there any ambiguity in the ordering of operators, as is expressed by the term with the arbitrary coefficient B. This is understood in terms of the ambiguity in the quantum version of (3.8). In fact demanding consistency of (5.2) with (5.1) we obtain

$$\pi_i=P_i-\frac{\mathrm{i}}{2}\frac{\partial h_{ml}}{\partial A^i}\bar{\psi}^l{}_\alpha\psi^m{}_\alpha+\mathrm{i}B\Gamma^m_{mi}. \tag{5.3}$$

A consistent formalism can be achieved with an arbitrary coefficient B, but for simplicity, we adopt the natural choice $B=0$.

Finally for the set of variables defined by (4.1) and (4.4) we have without ambiguity the following non-vanishing commutators and anticommutators

$$[R_i, A^j]=[\bar{R}_i, \bar{A}^j]=-\mathrm{i}\delta^j_i, \qquad \{\lambda^a{}_\alpha, \bar{\lambda}^b{}_\beta\}=\eta^{ab}\delta_{\alpha\beta}. \tag{5.4a, b}$$

6. The quantum mechanical supercharges and Hamiltonian

The supersymmetry algebra in the quantum theory is

$$\{Q_\alpha, Q_\beta\}=\{\bar{Q}_\alpha, \bar{Q}_\beta\}=0, \qquad \{Q_\alpha, \bar{Q}_\beta\}=2H\delta_{\alpha\beta}, \tag{6.1a, b}$$

where H is the operator ordered Hamiltonian.

The operator ordering problem for the supercharges and H by virtue of (6.1b) is resolved uniquely by demanding that they satisfy the above algebra and have the correct classical limits. The following results are obtained for the three sets of variables introduced above:

$$Q_\alpha=2^{1/2}\varepsilon_{\alpha\beta}\left(P_i-\frac{\mathrm{i}}{2}\frac{\partial h_{mn}}{\partial A^i}\bar{\psi}^n{}_\gamma\psi^m{}_\gamma\right)\psi^i{}_\beta \tag{6.2a}$$

$$H=\left(P_i-\frac{\mathrm{i}}{2}\frac{\partial h_{mn}}{\partial A^i}\bar{\psi}^n{}_\alpha\psi^m{}_\alpha\right)h^{ij}\left(\bar{P}_j+\frac{\mathrm{i}}{2}\frac{\partial h_{kl}}{\partial \bar{A}^j}\bar{\psi}^l{}_\beta\psi^k{}_\beta\right)+\tfrac{1}{4}k_{ijkl}\bar{\psi}^j{}_\alpha\varepsilon_{\alpha\beta}\bar{\psi}^l{}_\beta\psi^i{}_\gamma\varepsilon_{\gamma\delta}\psi^k{}_\delta, \tag{6.2b}$$

$$Q_\alpha=2^{1/2}\varepsilon_{\alpha\beta}\pi_i\psi^i{}_\beta, \qquad \bar{Q}_\alpha=2^{1/2}\varepsilon_{\alpha\beta}\bar{\psi}^i{}_\beta\bar{\pi}_i, \tag{6.3a}$$

$$H=\pi_i h^{ij}\bar{\pi}_j+\tfrac{1}{4}k_{ijkl}\bar{\psi}^j{}_\alpha\varepsilon_{\alpha\beta}\bar{\psi}^l{}_\beta\psi^i{}_\gamma\varepsilon_{\gamma\delta}\psi^k{}_\delta, \tag{6.3b}$$

$$Q_\alpha=2^{1/2}\varepsilon_{\alpha\beta}(R_i-\mathrm{i}\omega_{\bar{b}ia}\bar{\lambda}^b{}_\gamma\lambda^a{}_\gamma)e^i_d\lambda^d{}_\beta, \tag{6.4a}$$

$$H=(R_i-\mathrm{i}\omega_{\bar{b}ia}\bar{\lambda}^b{}_\gamma\lambda^a{}_\gamma)h^{ij}(\bar{R}_j+\mathrm{i}\bar{\omega}_{\bar{c}jd}\bar{\lambda}^c{}_\gamma\lambda^d{}_\gamma)+\tfrac{1}{4}k_{abcd}\bar{\lambda}^b{}_\alpha\varepsilon_{\alpha\beta}\bar{\lambda}^d{}_\beta\lambda^a{}_\gamma\varepsilon_{\gamma\delta}\lambda^c{}_\delta. \tag{6.4b}$$

In (6.4*b*) we have used the definition $k_{abcd}=e^i_a\bar{e}^j_b e^k_c\bar{e}^l_d k_{ijkl}$.

If the above expressions for the supercharges and Hamiltonian are to be compatible with each other we must have the following relationships between the three sets of operator variables:

$$P_i=\pi_i+\tfrac{1}{2}\mathrm{i}(\partial h_{mn}/\partial A^i)\bar{\psi}^n{}_\alpha\psi^m{}_\alpha, \tag{6.5}$$

$$P_i=R_i+\tfrac{1}{2}\mathrm{i}(\eta_{db}e^d_{k,i}e^k_a-\bar{e}^k_b\bar{e}^d_{k,i}\eta_{ad})\bar{\lambda}^b{}_\alpha\lambda^a{}_\alpha, \tag{6.6}$$

$$\pi_i=R_i-\mathrm{i}\omega_{\bar{b}ia}\bar{\lambda}^b{}_\alpha\lambda^a{}_\alpha. \tag{6.7}$$

These equations are deduced by either demanding consistency with the commutation relationships (given in § 5) satisfied by the variables occurring in them or, more easily, by inspection of (6.2), (6.3) and (6.4).

It should be remarked that the results of this section reflect the choice ($B=0$) made in connection with (5.2).

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References

Alvarez-Gaume L and Freedman D Z 1980 *A Simple Introduction to Complex Manifolds* in *Unification of the Fundamental Interactions* eds S Ferrara *et al* (New York: Plenum) p 41

—— 1981 *Commun. Math. Phys.* **80** 433

Casalbuoni P 1976 *Nuovo Cimento* **33A** 133, **34A** 389

Davis A C, Macfarlane A J, Popat P and van Holten J W 1984 *J. Phys. A. Math. Gen.*, **17** 2945

Dirac P A M 1964 *Lectures on Quantum Mechanics, Belfer Graduate School, New York*

Flaherty E J 1976 *Hermitian and Kahlerian Geometry in Relativity* (Berlin: Springer)

Zumino B 1979 *Phys. Lett.* **87B** 203